Some Stabbing Problems of Line Segments Solved with Linear Programming

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1 Introduction

In this paper we introduce a class of stabbing problems that can be solved using linear programming in $O(n)$ time. We start addressing the following:

Problem 1. Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in the plane. Suppose that the elements of P start moving vertically at time $t = 0$ and at the same speed v. As p_i moves up, at time t the point p_i has traversed a line segment l_t^i of length $t \cdot v$, starting at p_i, let us denote as $p_i(t) = p_i + t \cdot v$. Our problem is to find the smallest t such that there exist a line ℓ that stabs $l_t^1, l_t^2, \cdots, l_t^n$, see Figure 1a. We prove that this problem can be solved in $O(n)$ time.

We also address the following variations to our problem:

Problem 2. Each point p_i moves vertically at its own speed v_i.

Problem 3. Each point p_i moves at its own direction s_i and at its own speed v_i.

Problem 4. Same problems as above for $p_i \in \mathbb{R}^d$ where d is fixed.

We will show that all of the above problems can be solved using linear programming in $2d - 1$, and thus can be solved in $f(d) \times n$ time, which is linear time for constant d.

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In Section 2, we define some concepts of linear programming and point to line transformations. In Section 3, we demonstrate that all the described problems can be solved in $O(n)$ time, when d is fixed.

2 Preliminaries

The problem of geometric separability of two sets of points R and B in $\mathbb{R}^d$ is to decide if there is a hyperplane that leaves all of the elements of B in one of the open semiplanes determined by the hyperplane, and all of the elements of R in the other. It is well known that a linear programming problem with d dimension and n variables can be solved in $O(n)$ time when d is fixed [2].

The dual of a point $p = (a, b)$ of the plane, denoted by ℓ_p, is the non-vertical line with equation $y = ax + b$. The dual of ℓ_p is p. Recall that in the dual plane the *lower envelope* is the boundary of the intersection of the halfplanes lying below the lines. Similarly, the *upper envelope* is formed by considering the intersection of the halfplanes lying above the lines.

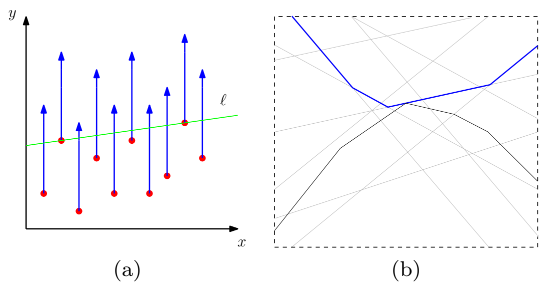

Figure 1: *a)* Set of n points in the plane moving vertically at the same speed. *b)* Dual plane showing the intersection of the upper and lower envelopes.

3 Stabbing line segments

In this section we describe our algorithm to obtain the smallest time t, if it exists, such that at time t there is a line ℓ that stabs all of the line segments l_t^i. Let P_r be the set of red points (resp. lines) containing $P = \{p_1, \dots, p_n\}$, and $P_b(t) = \{p_1(t), \dots, p_n(t)\}$. A transformation to the dual plane considering the time is given as follows: every point $p_i = (a_i, b_i + t)$ is mapped to the line $y = a_i x + b_i + t$. The elements of P_r are mapped to the lines $\mathcal{L}_r = \{a_i x + b_i \mid i = 1, \dots, n\}$. Similarly, the elements of P_b are mapped to the lines $\mathcal{L}_b = \{y = a_i x + b_i + t \mid i = 1, \dots, n\}$. We note that while points start moving in the primal plane their corresponding lines in the dual plane move upward. After sometime if a feasible region exists the upper envelope of $\mathcal{L}_b$ will intersect the lower envelope of $\mathcal{L}_r$ and that point would be the solution, see Figure 1b. $\mathcal{L}_r$ and $\mathcal{L}_b$ represent the below and above constraints, respectively. So $\mathcal{L}_r$ can be represented as $a_i x - y + b_i + t \leq 0$ and $\mathcal{L}_b$ can be represented as $a_i x - y + b_i \geq 0$. Finally our problem can be stated as a linear programming problem in $\mathbb{R}^3$ as follows:

$$\begin{aligned} &\text{minimize } t \\ &\text{subject to } a_i x - y - b_i \leq 0 \\ &\qquad a_i x - y - (b_i + t) \geq 0 \end{aligned}$$

Thus using Meggido's partition algorithm [1], the linear programming problem is solved in $O(n)$ time.

Thus we have the following result:

Theorem 1. *The smallest time t such that a line ℓ stabs the line segments l_t^i can be calculated in $O(n)$ time.*

Let us consider now Problem 2. The set of lines $\mathcal{L}_r$ does not change and their upper envelope remains the same, however now the set of lines $\mathcal{L}_b$ move upwards at different speeds and the lower envelope changes over the time. To solve these new constraints we asociate the speeds as follows: for every $p_i = (a_i, b_i + v_i \cdot t)$, the line $y = a_i x + b_i + v_i \cdot t$ is mapped. Then $\mathcal{L}_b = \{y = a_i x + b_i + v_i \cdot t \mid i = 1, \dots, n\}$. Problem 2 can be stated as the following linear programming problem in $\mathbb{R}^3$:

$$\begin{aligned} &\text{minimize } t \\ &\text{subject to } a_i x - y - b_i \leq 0 \\ &\qquad a_i x - y - (b_i + v_i \cdot t) \geq 0 \end{aligned}$$

For Problem 3. The set of lines $\mathcal{L}_r$ remains the same, for the case of $\mathcal{L}_b$ we associate the *inclination* of every point as follows: every $p_i = (a_i + s_i, b_i + v_i \cdot t)$ is mapped to the line $y = (a_i + s_i)x + b_i + v_i \cdot t$ then $\mathcal{L}_b = \{y = (a_i + s_i)x + b_i + v_i \cdot t \mid i = 1, \dots, n\}$. Finally Problem 3 can be stated as the following linear programming problem in $\mathbb{R}^3$:

$$\begin{aligned} &\text{minimize } t \\ &\text{subject to } a_i x - y - b_i \leq 0 \\ &\qquad (a_i + s_i)x - y - (b_i + v_i \cdot t) \geq 0 \end{aligned}$$

Consider the Problem 4 for points in $\mathbb{R}^3$. The points move vertically at different speeds, now the transformation to the dual space is defined as follows: every point $p_i = (a, b, c + v_i \cdot t)$ is mapped to the plane $z = a_i x + b_i y + c_i + v_i \cdot t$. The below constraints are defined as $a_i x + b_i y - z + c_i \geq 0$ while the above constraints $a_i x + b_i y - z + c_i + v_i \cdot t \leq 0$. Problem 4 can be defined as the following linear programming problem in $\mathbb{R}^4$:

$$\begin{aligned} &\text{minimize } t \\ &\text{subject to } a_i x + b_i y - z - c_i \leq 0 \\ &\qquad a_i x + b_i y - z - (c_i + v_i \cdot t) \geq 0 \end{aligned}$$

The d-dimensional case can be solved in linear time, for lake of space we do not give more details but we enunciate the following theorem.

Theorem 2. *For any fixed dimension d, Problems 2, 3, and 4 can be solved in $O(n)$ time.*

References

[1] Nimrod Megiddo. Linear-time algorithms for linear programming in $\mathbb{R}^3$ and related problems. *SIAM Journal on Computing*, 12(4):759–776, 1983.

[2] Nimrod Megiddo. Linear programming in linear time when the dimension is fixed. *Journal of the ACM*, 31(1):114–127, 1984.